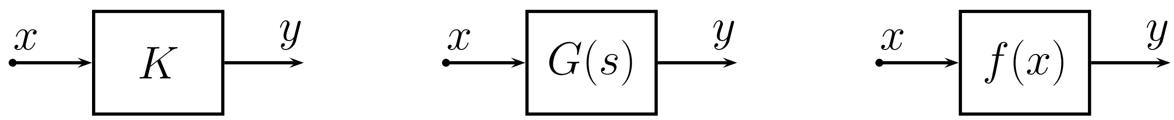
Reduction of block diagrams

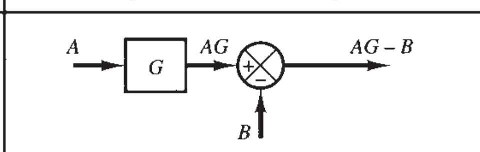
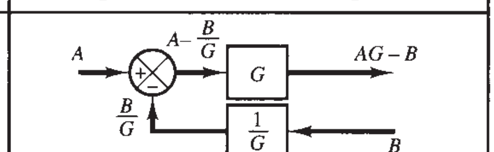
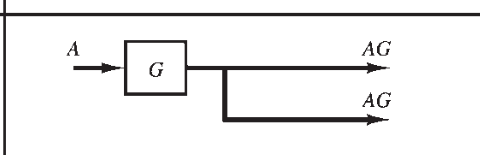
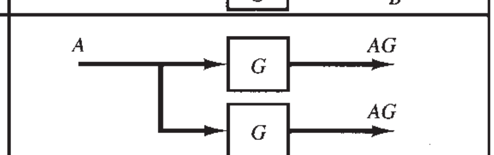
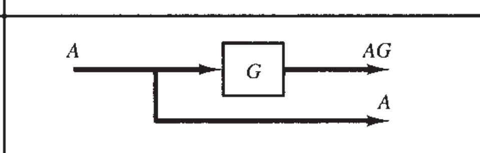
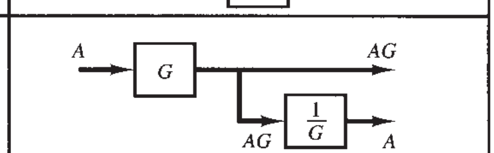
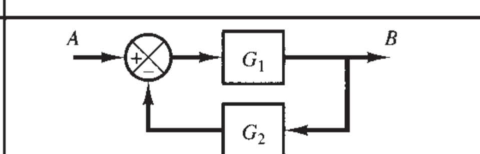
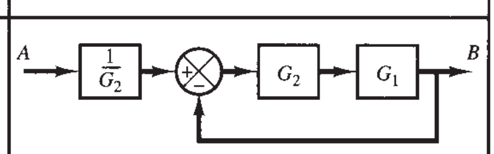
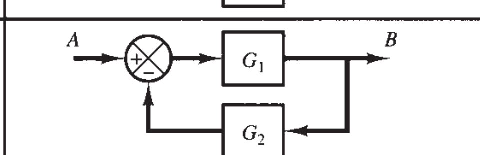
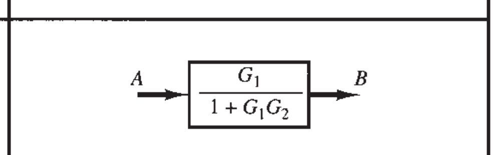
• Complex systems are often graphically represented by *block diagrams* obtained connecting in series/parallel the oriented blocks (static, dynamic, linear, non-linear, etc.) which describe the functionalities of the physical elements which are present in the system.



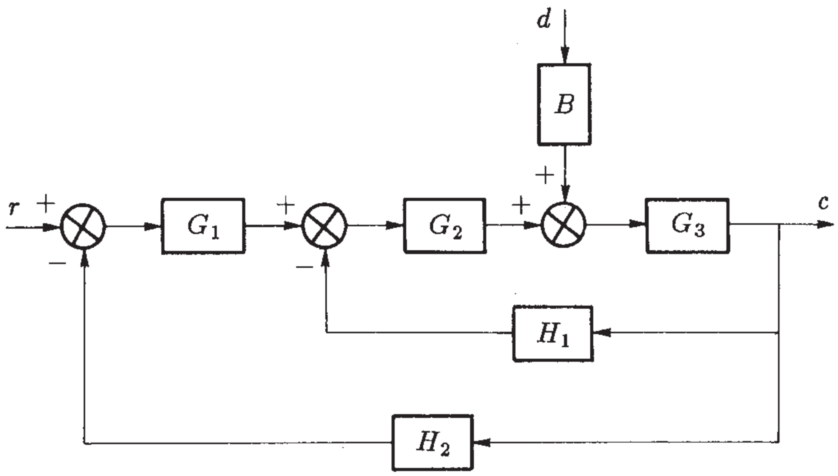
$y(t) = K\,x(t)$	$Y(s) = G(s)\,X(s)$	$y = f(x)$
1) Static linear block	2) Dynamic linear block	3) Static nonlinear block

• In the block diagrams, the individual oriented elements are connected to each other by "*branch points*" and "*summation points*":

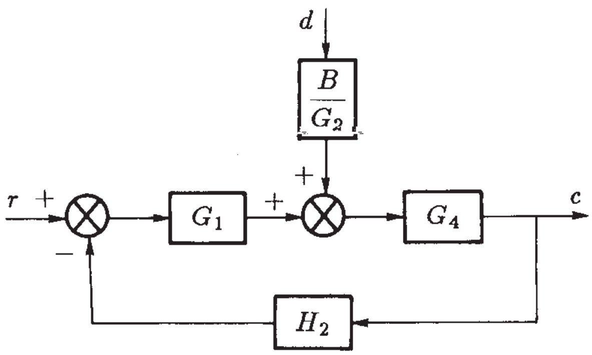
• Main rules for a graphical reduction of the block schemes:

	Original Block Diagrams	Equivalent Block Diagrams
1		
2		
3		
4		
5		

Example of graphical reduction of a block scheme

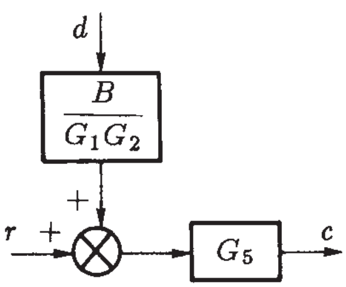


a)



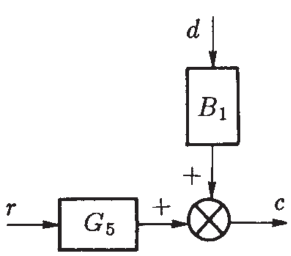
b)

$$G_4 = \frac{G_2G_3}{1+G_2G_3H_1}$$



c)

$$G_5 = \frac{G_1G_4}{1+G_1G_4H_2}$$



d)

$$B_1 = \frac{B\,G_5}{G_1G_2}$$

- Minimum form:

$$c = \frac{G_1G_2G_3\,r + B\,G_3\,d}{1+G_2G_3H_1+G_1G_2G_3H_2}$$

Mason's formula

- Given a block scheme, an input X and an output Y, the Mason's formula is a simple and direct way for computing the *transfer function* $G = \frac{Y}{X}$ that links the input X to the output Y:

$$G = \frac{Y}{X} = \frac{1}{\Delta} \sum_{i \in \mathcal{P}} P_i \Delta_i$$

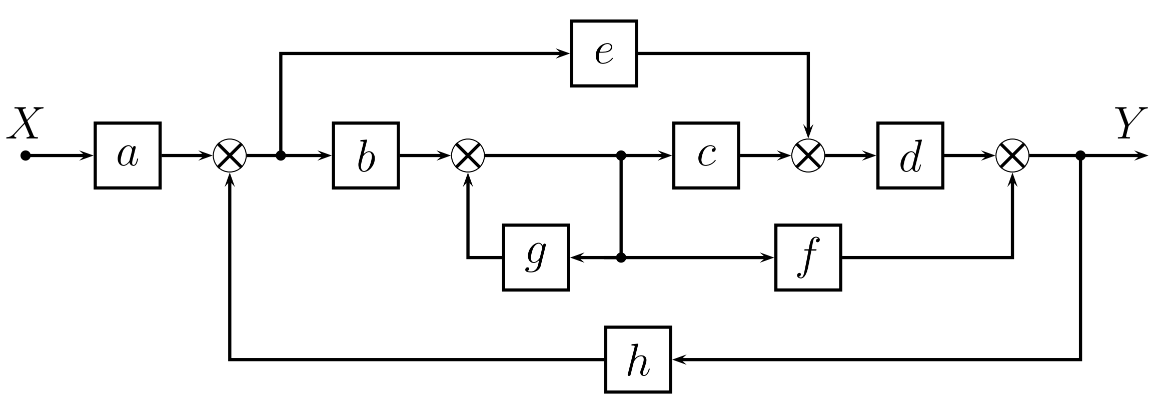
- $\mathcal{P}$ is the set of indices of all the distinct paths that connect the input X to the output Y. P_i is the coefficient of the i-th path, that is the product of the coefficients of all the elements which belongs to the i-th path. Δ is the determinant of the whole block diagram. Δ_i is the determinant of the partial block diagram that is obtained by eliminating from the scheme all the elements belonging to the i-th path.

- The determinant Δ of a block diagram is calculated as follows:

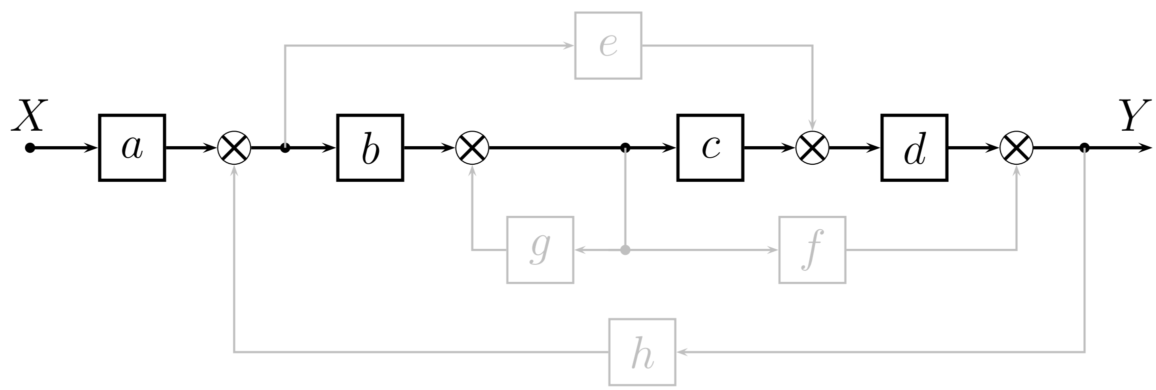
$$\Delta := 1 - \sum_{i \in \mathcal{J}_1} A_i + \sum_{(i,j) \in \mathcal{J}_2} A_i A_j - \sum_{(i,j,k) \in \mathcal{J}_3} A_i A_j A_k + \ldots$$

where A_i is the coefficient of the i-th ring (i.e. a closed path), $\mathcal{J}_1$ is the set of indices of all the rings of the block diagram, $\mathcal{J}_2$ is the set of indices of all the pairs of rings that do not touch each other, ..., $\mathcal{J}_n$ is the set of indices of all the n-ple of rings that do not touch n to n.

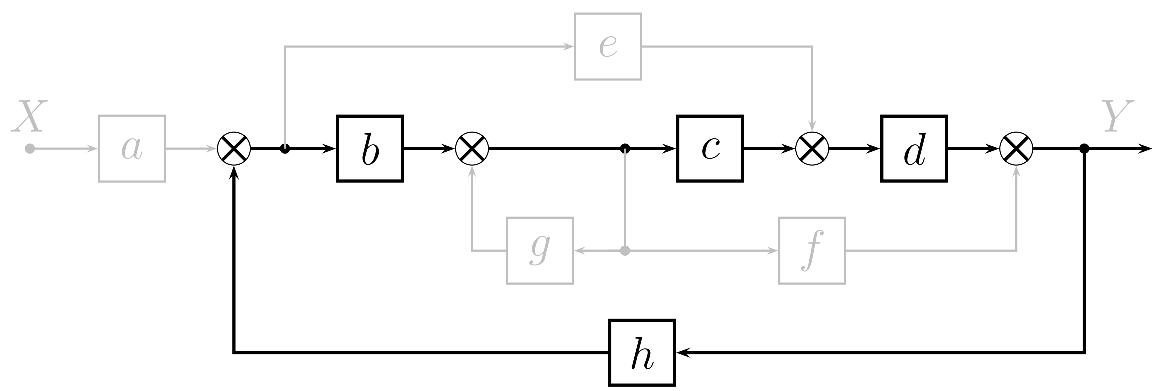
- Example. Given the following block diagram, calculate the transfer function $G = \frac{Y}{X}$ that links the input X to the output Y:



• A **path** is a sequence of adjacent branches and nodes without rings in which each element is crossed only once. The **coefficient** P of the path is the product of the gains of the branches that compose the path. Example: the coefficient P_1 of the path highlighted in the following figure is $P_1 = abcd$.



• A **ring** is a closed path. The **coefficient** A of the ring is the product of the gains of the branches that compose the ring. Example: the coefficient A_2 of the ring highlighted in the following figure is $A_2 = bcdh$.



• Two paths or two rings **do not touch** each other when they have no common points.

• To calculate the determinant Δ of a block diagram, it is necessary to compute the $\mathcal{P}$, $\mathcal{J}_1$, $\mathcal{J}_2$ sets, etc.

• The set $\mathcal{P} = \{1, 2, 3\}$ is the set of indices of all the paths of the block diagram that connect the input variable X to the output variable Y. For each index i, the corresponding path coefficient P_i must be computed:

$$P_1 = abcd, \qquad P_2 = aed, \qquad P_3 = abf.$$

• The set $\mathcal{J}_1 = \{1, 2, 3, 4\}$ is the set of indices of all the rings in the block diagram. For each index i, the corresponding ring coefficient A_i must be computed:

$$A_1 = edh, \qquad A_2 = bcdh, \qquad A_3 = bfh, \qquad A_4 = g.$$

• The set $\mathcal{J}_2 = \{(1, 4)\}$ is the set of COUPLES of indexes of the rings of the block diagram that DO NOT touch each other:

$$\mathcal{J}_2 = \{(1, 4)\}.$$

• The set $\mathcal{J}_n = \{\ \}$ for $n \in [3,\ 4,\ \ldots]$ is the set of n-PLES of ring indices of the block diagram that DO NOT touch n to n:

$$\mathcal{J}_3 = \mathcal{J}_4 = ... = \mathcal{J}_n = \{\ \}.$$

• Once the sets $\mathcal{J}_1$, $\mathcal{J}_2$, ..., $\mathcal{J}_n$ and the coefficients A_i of all rings have been calculated, the determinant Δ the block diagram can be obtained as follows:

$$\Delta \overset{def}{=} 1 - \sum_{i \in \mathcal{J}_1} A_i + \sum_{(i,j) \in \mathcal{J}_2} A_i A_j - \sum_{(i,j,k) \in \mathcal{J}_3} A_i A_j A_k + ...$$

For the considered case we have that:

$$\sum_{i \in \mathcal{J}_1} A_i = edh + bcdh + bfh + g, \qquad \sum_{(i,j) \in \mathcal{J}_2} A_i A_j = edhg$$

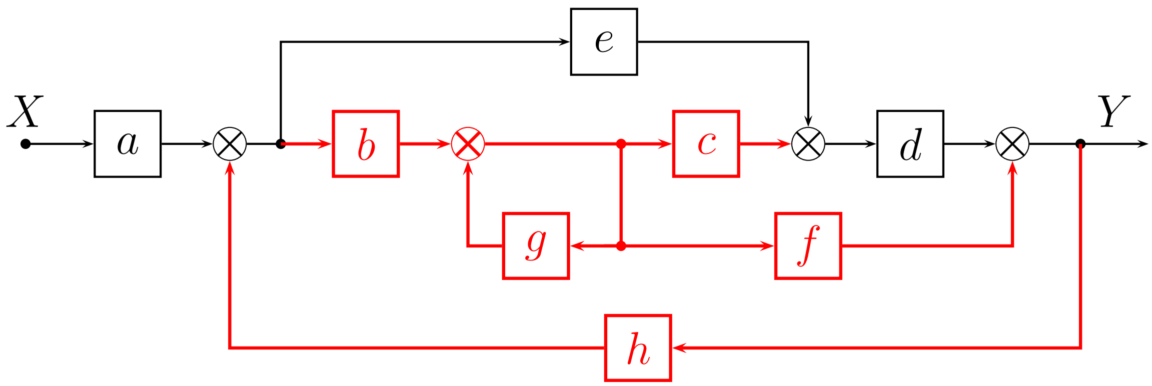
so the determinant Δ of the block diagram is:

$$\Delta = 1 - edh - bcdh - bfh - g + edhg.$$

Remarks:

- The determinant of a block scheme depends ONLY on the rings which are present inside the block scheme and not on the input and output variables.
- All the possible transfer functions that can be obtained from a block diagram are characterized by the same determinant Δ.
- The determinants Δ_i of the partial block schemes associated to the paths P_i are calculated in the same way.

- The partial block diagram associated with the path $P_2 = aed$, for example, is determined by deleting all nodes and all branches belonging to the path P_2. For the following partial block diagram we have: $\Delta_2 = 1 - g$.



- For the considered system, the Δ_i determinants of the partial block schemes associated with the P_i paths are as follows:

$$\Delta_1 = 1, \qquad \Delta_2 = 1 - g, \qquad \Delta_3 = 1.$$

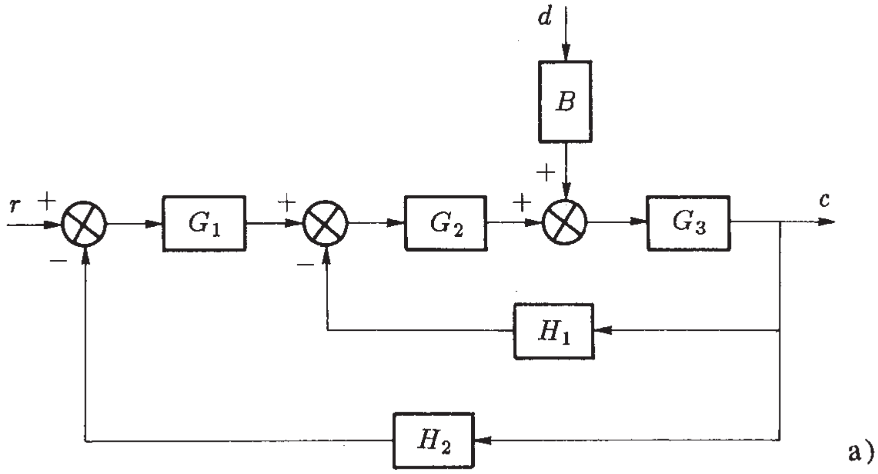
- The numerator of Mason's formula is therefore the following:

$$\sum_{i \in \mathcal{P}} P_i \Delta_i = abcd(1) + aed(1 - g) + abf(1)$$

- The transfer function $G(s) = \frac{Y(s)}{X(s)}$ which links the input X to the output Y is then the following:

$$G(s) = \frac{abcd + aed(1 - g) + abf}{1 - edh - bcdh - bfh - g + edhg}$$

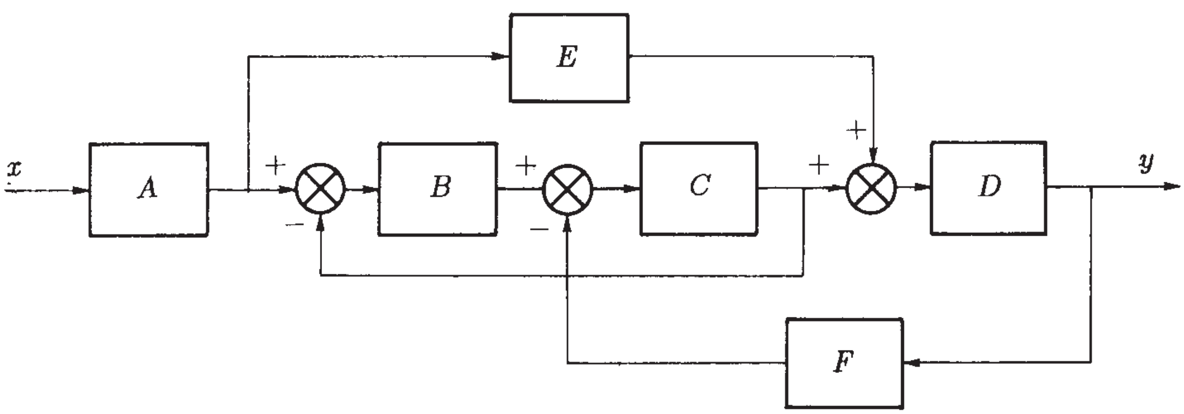
- **Example 1**:



Minimum form:

$$c = \frac{G_1 G_2 G_3\, r + B\, G_3\, d}{1 + G_2 G_3 H_1 + G_1 G_2 G_3 H_2}$$

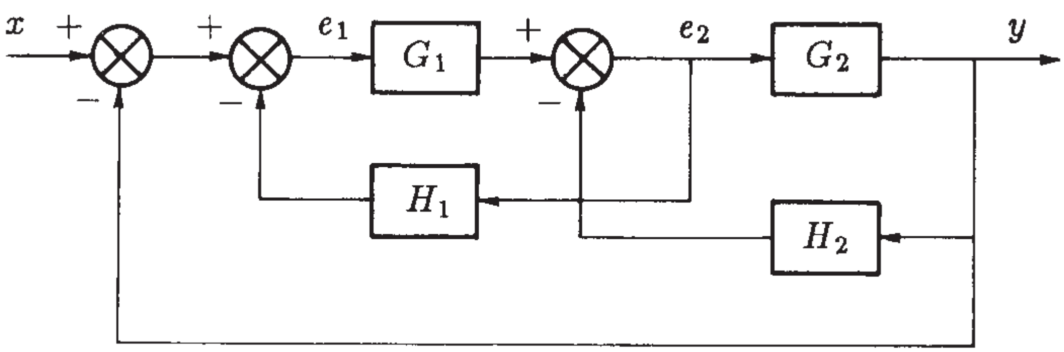
- **Example 2**:



- Transfer function:

$$\frac{y}{x} = \frac{A\,D\,B\,C + A\,D\,E\,(1{+}B\,C)}{1 + B\,C + C\,D\,F}$$

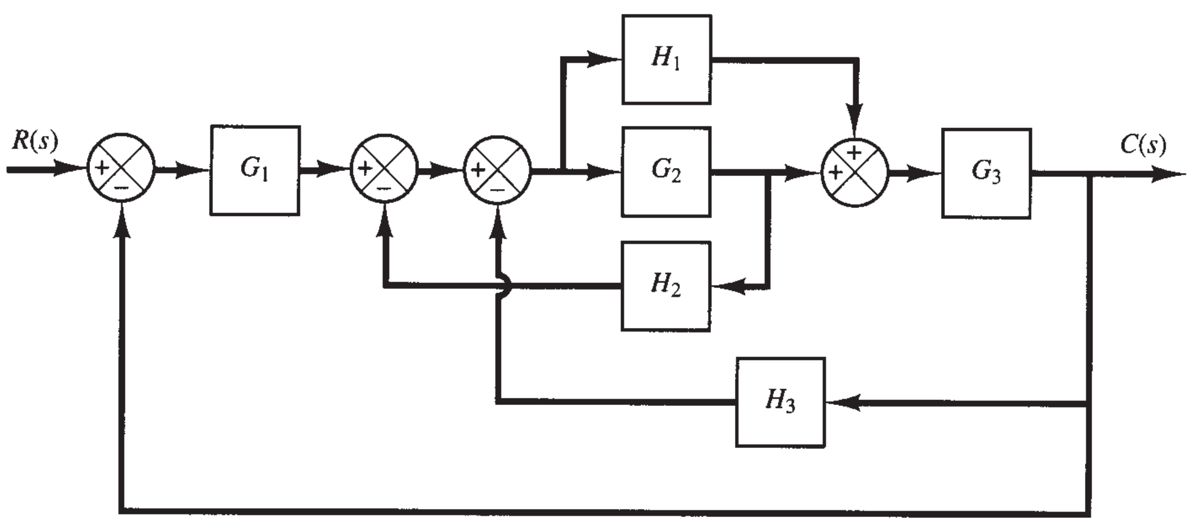
- **Example 3**:



- Transfer function:

$$\frac{y}{x} = \frac{G_1\,G_2}{1 + G_1\,H_1 + G_2\,H_2 + G_1\,G_2}$$

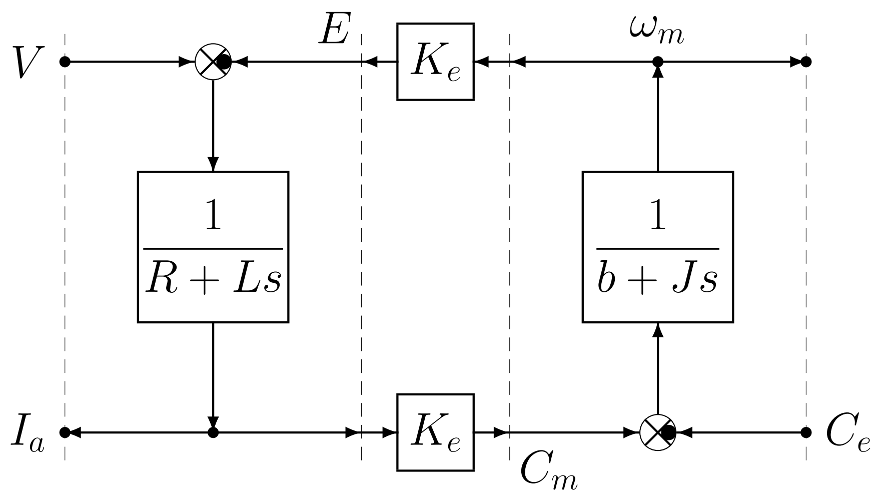
- **Example 4**:



- Transfer function:

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3 + G_1H_1G_3}{1 + G_1G_2G_3 + G_1H_1G_3 + G_2H_2 + G_2G_3H_3 + H_1G_3H_3}$$

- **Example 5.** Block diagram of a DC electric motor:



- The output variable $\omega_m(s)$ can be expressed as a function of the input variables $V(s)$ and $C_e(s)$ as follows:

$$\omega_m(s) = G_1(s)\,V(s) + G_2(s)\,C_e(s)$$

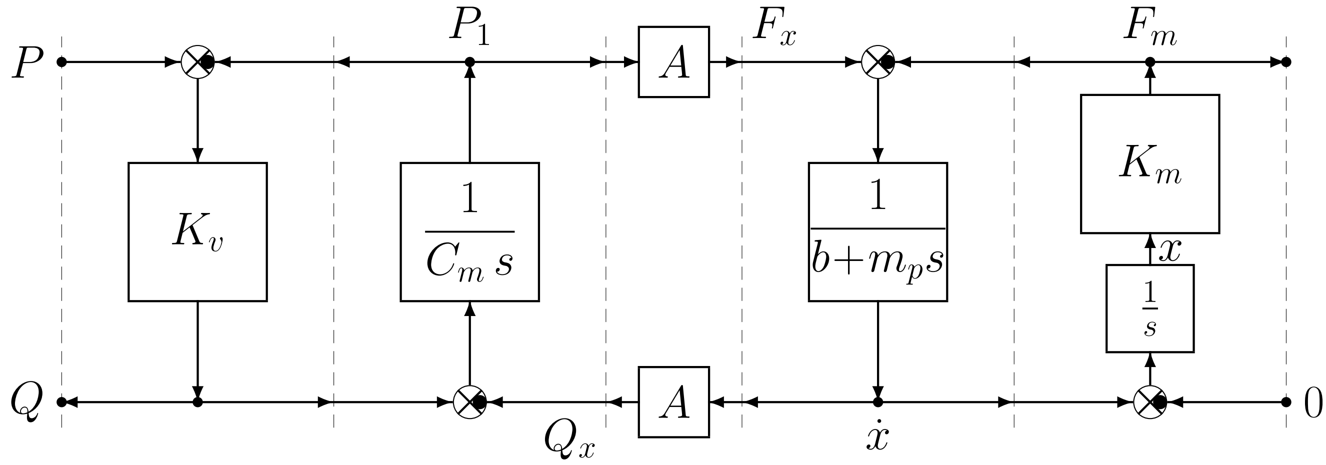
where $G_1(s)$ links the input $V(s)$ to the output $\omega_m(s)$:

$$G_1(s) = \frac{\omega_m(s)}{V(s)} = \frac{\dfrac{K_e}{(R+L\,s)(b+J\,s)}}{1+\dfrac{K_e^2}{(R+L\,s)(b+J\,s)}} = \frac{K_e}{(R+L\,s)(b+J\,s)+K_e^2}$$

and $G_2(s)$ links the input $C_e(s)$ to the output $\omega_m(s)$:

$$G_2(s) = \frac{\omega_m(s)}{C_e(s)} = \frac{-\dfrac{1}{(b+J\,s)}}{1+\dfrac{K_e^2}{(R+L\,s)(b+J\,s)}} = \frac{-(R+L\,s)}{(R+L\,s)(b+J\,s)+K_e^2}$$

- **Example 5.** Block diagram of an hydraulic clutch:



Using the Mason's formula and the following auxiliary variables:

$$G_1 = K_v, \qquad G_2 = \frac{1}{C_m s}, \qquad G_3 = \frac{1}{b + m_p s}, \qquad G_4 = \frac{K_m}{s}$$

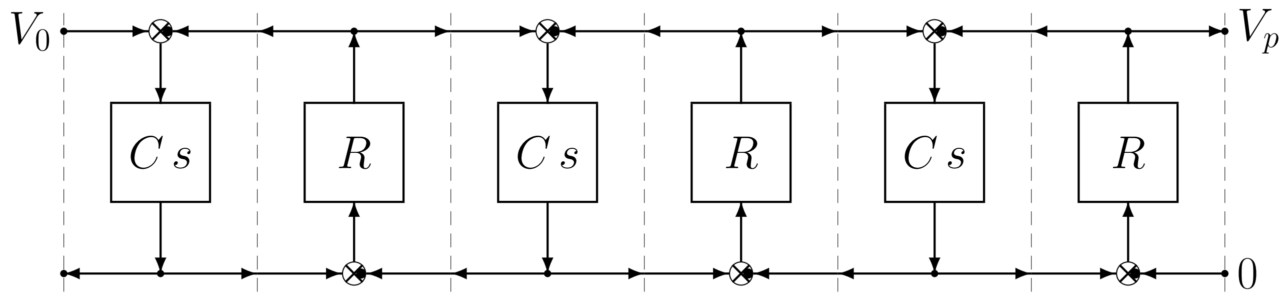
one can easily obtain the following system transfer function $G(s)$:

$$G(s) = \frac{F_m(s)}{P(s)} = \frac{A\, G_1\, G_2\, G_3\, G_4}{1 + G_1\, G_2 + A^2\, G_2\, G_3 + G_3\, G_4 + G_1\, G_2\, G_3\, G_4}$$

Replacing the auxiliary variables one obtains:

$$G(s) = \frac{A K_m K_v}{C_m m_p s^3 + (C_m\, b + K_v m_p) s^2 + (A^2 + C_m K_m + K_v\, b) s + K_m K_v}$$

- **Example 6.** Consider the following block diagram:

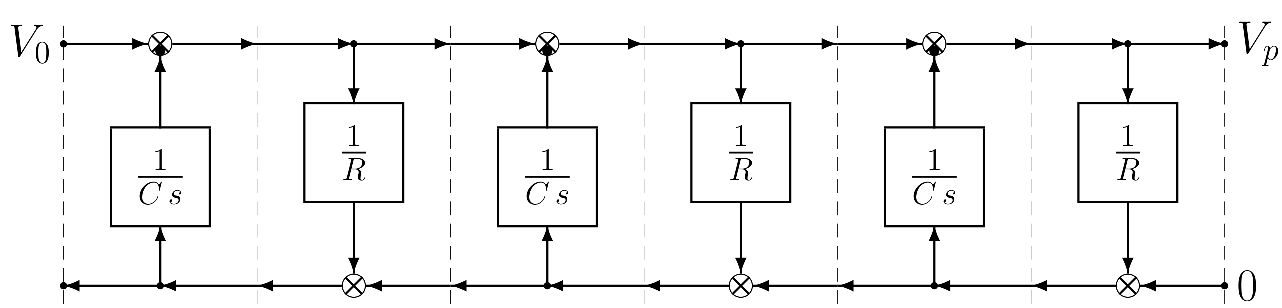


The transfer function $G(s)$ which links the input $V_0(s)$ to the output $V_p(s)$ can be easily obtained using the Mason's formula:

$$G(s) = \frac{V_p(s)}{V_0(s)} = \frac{R^3C^3s^3}{1 + 5\,R\,C\,s + 6\,R^2\,C^2\,s^2 + \,R^3\,C^3\,s^3}$$

In fact, within the block diagram there are 5 distinct rings, all having ring gains $-RCs$. Moreover, there are 6 couples of rings that do not touch each other, and one set of rings that do not touch three to three. The only path that goes from V_0 to V_p pass through all the blocks.

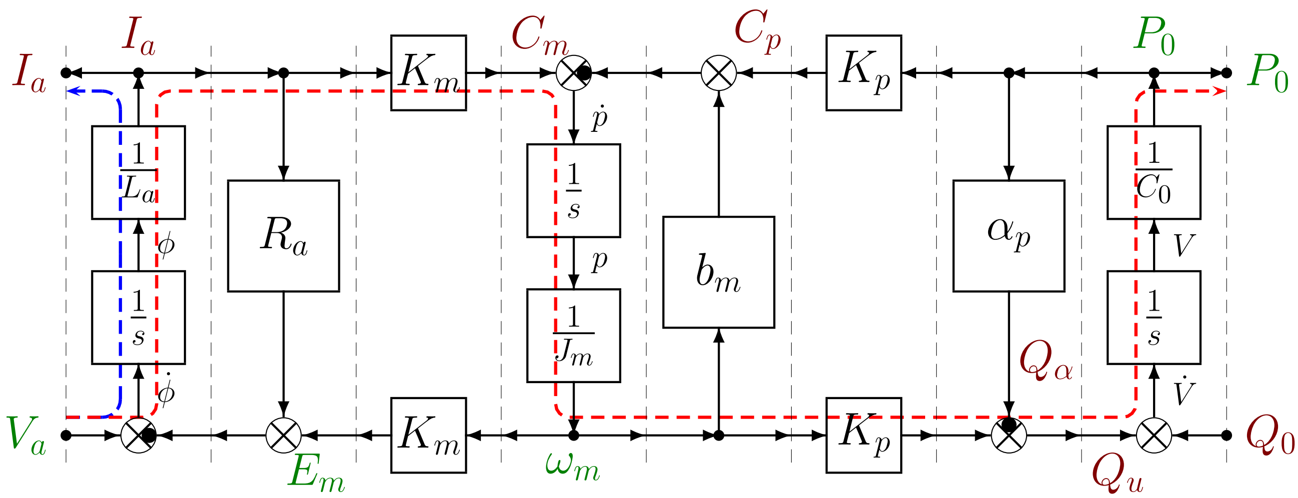
The previous block scheme (which is not physically realizable) is equivalent to the following physically realizable block scheme:



One can easily verify that applying the Mason's formula to this block scheme one obtains the same transfer function $G(s)$ obtained applying the Mason's formula to the previous block scheme.

Relative degree of a transfer function $G(s)$

- Let us consider a generic block scheme:



- For each transfer function $G(s) = \frac{Y(s)}{U(s)}$ which links an input $u(t)$ to the output $y(t)$, the following properties hold:

  1) the order of function $G(s)$ is equal to the number n of independent dynamic elements which store energy within the system;

  2) the poles of function $G(s)$ are equal to the solutions of equation $\Delta(s) = 0$ where $\Delta(s)$ is the determinant of the block scheme;

  3) the **relative degree** of function $G(s)$ is equal to the minimum number r of integrators present in all the paths that link the input $u(t)$ to the output $y(t)$;

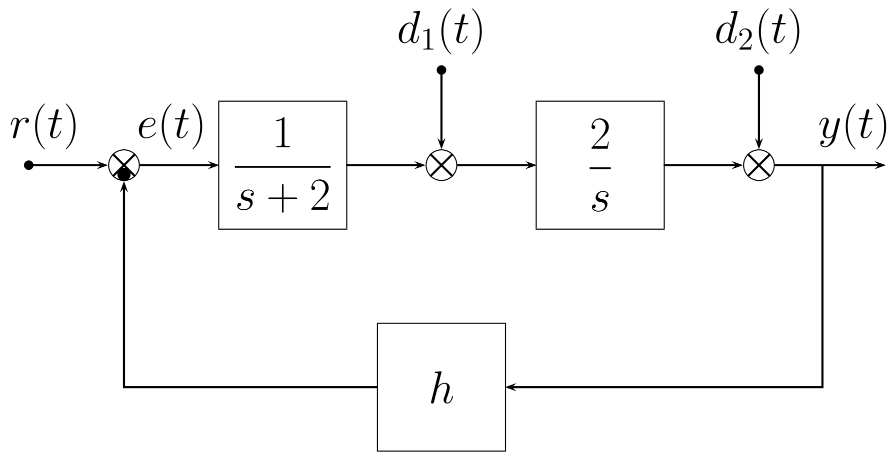
  4) if there is only one path $\mathcal{P}_1$ that links the input $u(t)$ to the output $y(t)$, then the zeros of function $G(s)$ are equal to the solutions of equation $\Delta_1(s) = 0$ where $\Delta_1(s)$ is the determinant of the reduced block scheme obtained from the original one eliminating all the blocks touched by path $\mathcal{P}_1$;

  $G(s) = \frac{P_0}{V_a}$ has 3 poles and 0 zeros because the relative degree is $r = 3$;

  $G(s) = \frac{I_a}{V_a}$ has 3 poles and 2 zeros because the relative degree is $r = 1$;

- Note: the higher is the relative degree the more difficult is the control.

- **Example 7.** Let us consider the following feedback system:



- Calculate the steady-state value of the variable $e(t)$ in the presence of the following signals: $r(t) = t$, $d_1(t) = 1$ and $d_2(t) = 1$.

  Solution. Using the Laplace transform and the linearity property of the system one obtains:

$$\begin{aligned} E(s) &= \frac{R(s) - \frac{2h}{s}D_1(s) - hD_2(s)}{1 + \frac{2h}{s(s+2)}} \\ &= \frac{s(s+2)R(s) - 2h(s+2)D_1(s) - hs(s+2)D_2(s)}{s^2 + 2s + 2h} \end{aligned}$$

  Being $R(s) = \frac{1}{s^2}$ and $D_1(s) = D_2(s) = \frac{1}{s}$, you have that:

$$E(s) = \frac{(s+2) - 2h(s+2) - hs(s+2)}{s(s^2 + 2s + 2h)} = \frac{(s+2)(1 - 2h - hs)}{s(s^2 + 2s + 2h)}$$

  Applying the final value theorem one obtains:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{1 - 2h}{h} = \frac{1}{h} - 2$$