Technical Report: Proof of Bayesian Cramer Rao Bound for Localization Maps

Francesco Montorsi*[1], Santiago Mazuelas†[2], and Giorgio M. Vitetta‡[1]

[1]Department of Information Engineering, University of Modena and Reggio Emilia
[2]Laboratory for Information and Decision Systems (LIDS), Massachusetts Institute of Technology (MIT)

December 2, 2011

1 Introduction

In this Technical Report we provide the proof of the Bayesian Cramer Rao Bound (BCRB) for Localization Maps published in [1]. The notation used in the following is introduced in Sec. II of that document.

The starting point for our computations is the Bayesian Fisher Information Matrix (BFIM) decomposed as [2, p. 183, eq. (75)]:

$$\mathbf{J} = \mathbf{J}_{\mathbf{z}|\mathbf{p}} + \mathbf{J}_{\mathbf{p}}, \tag{1}$$

where

$$\mathbf{J}_{\mathbf{z}|\mathbf{p}} \triangleq \mathbb{E}_{\mathbf{z},\mathbf{p}} \left\{ -\frac{\partial}{\partial \mathbf{p}} \left[\frac{\partial}{\partial \mathbf{p}} \ln f(\mathbf{z}|\mathbf{p}) \right]^T \right\}$$

*francesco.montorsi@unimore.it
†mazuelas@mit.edu
‡giorgio.vitetta@unimore.it

and

$$\mathbf{J_p} \triangleq \mathbb{E}_{\mathbf{p}} \left\{ -\frac{\partial}{\partial \mathbf{p}} \left[\frac{\partial}{\partial \mathbf{p}} \ln f(\mathbf{p}) \right]^T \right\} \tag{2}$$

are respectively the contribution originating from the noisy data vector $\mathbf{z}$ and that due to *a-priori* information, respectively.

2 Derivation of the Bounds for Localization

2.1 Derivation of the BFIM for Uniform Maps with Generic-Shape

We start the derivation for the BFIM for uniform maps with a generic shape introducing some properties for the smoothing function mentioned in [1, Sec II.B] which allows us to model the map pdf. Then the 1-D result is obtained and extended to the 2-D case.

2.1.1 Modeling of the Pdf

Let $s(t)$ be a continuous and differentiable function $s : \mathbb{R} \to \mathbb{R}$. Assume that $s(t)$ has the following additional properties:

1. $s(t) \geq 0 \, \forall t \in \mathbb{R}$;
2. $\int s(t)dt = 1$;
3. $\int \frac{\partial s(t)}{\partial t} dt = 0$;
4. $s(t)$ has support $\left[-\frac{1}{2}; +\frac{1}{2}\right]$;
5. $s(0) = 1$;

The function $s(\cdot)$ is then a pdf function (assumptions 1 and 2) and has an associated a-priori FI J_s (assumption 3 is the regularity condition that grants the FI existence) [3]. The assumptions 4 and 5 finally assure that $s(\cdot)$ can be used to model bounded statistical distributions i.e. maps defined as statistical distributions of the position to estimate [1], eventually with some scaling and translation.

A function $s(\cdot)$ satisfying the conditions above is dubbed in the following as "smoothing function". Examples of functions that satisfy those hypotheses are:

1. $s(t) = g\left(t+\frac{1}{2};\delta\right) g\left(-t+\frac{1}{2};\delta\right)$ where $g(t;\delta) \triangleq \frac{f(t+\delta)}{f(t+\delta)+f(-t+\delta)}$ and $f(t) \triangleq e^{-\frac{1}{t}}\mathrm{u}(t)$;

2. $s(t) = g\left(t+\frac{1}{2};\delta\right) g\left(-t+\frac{1}{2};\delta\right)$ where

$$g(t;\delta) \triangleq \begin{cases} 0 & t < -\delta \\ -\frac{t^3}{4\delta^3} + \frac{3t}{4\delta} + \frac{1}{2} & -\delta \le t \le +\delta \\ 1 & t > +\delta \end{cases}$$

Note that in the examples above $\delta \in \left(0; \frac{1}{2}\right]$ is a parameter which defines the steepness of the pdf $s(t)$, so that $\lim_{\delta\to 0} g\left(t;\delta\right) = \mathrm{u}(t)$, where $\mathrm{u}(t)$ is the unit step function (which however does not have an associated FI). In the latter example, the associated FI is easy to compute in closed form: $\mathrm{J}_s(\delta) = \frac{9\ln 3}{4\delta}$.

Finally note that $\tilde{s}(t,a,b) \triangleq \frac{1}{b} s\left(\frac{t-a}{b}\right)$ is still a pdf function and its associated FI is (see Eq. (2)):

$$\tilde{\mathrm{J}}_s \triangleq \mathbb{E}_t\left\{\left(\frac{\partial \ln}{\partial t}\frac{1}{b} s\left(\frac{t-a}{b}\right)\right)^2\right\} = \mathbb{E}_t\left\{\left(\frac{\partial \ln s\left(u\right)}{\partial u}\frac{1}{b}\right)^2\right\} = \frac{\mathrm{J}_s}{b^2}$$

2.1.2 Derivation of the BFIM for 1-D Maps

Consider a 1-D smoothed uniform map $f(x)$ with support $\mathcal{R} \subset \mathbb{R}$; 1-D uniform maps can always be modeled by a set of N_r disjoint 1-D rectangles centered in the points $\{x_n\}$ with widths $\{w_n\}$, where $n = 1, ..., N_r$. Thus the map pdf can be expressed as:

$$f(x) = \frac{1}{\mathcal{W}}\sum_{n=1}^{N_r} s\left(\frac{x-x_n}{w_n}\right) = \frac{1}{\mathcal{W}}\sum_{n=1}^{N_r} w_n \tilde{s}(x, x_n, w_n) \tag{3}$$

where $\mathcal{W} \triangleq \sum_{n=1}^{N_r} w_n$, $s(\cdot)$ and $\tilde{s}(\cdot)$ are the smoothing functions as defined in Appedix 2.1.1.

The a-priori FI associated with $f(x)$ is (see Eq.(2)) $\mathrm{J}_x \triangleq \mathbb{E}_x\left\{\left(\frac{\partial \ln f(x)}{\partial x}\right)^2\right\}$. To simplify the expression we consider that a) for each value of x there is only one rectangle at most for which $\left(\frac{\partial \ln f(x)}{\partial x}\right)^2 \neq 0$, and b) varying x over $\mathbb{R}$, all rectangles contribute to the FI integral. Thus the FI can be written as the sum of the FI contribute of each rectangle:

$$\mathrm{J}_x = \frac{1}{\mathcal{W}} \sum_{n=1}^{N_r} w_n \mathbb{E}_x \left\{ \left(\frac{\partial}{\partial x} \ln \tilde{s}(x, x_n, w_n) \right)^2 \right\}$$
$$= \frac{1}{\mathcal{W}} \sum_{n=1}^{N_r} w_n \tilde{\mathrm{J}}_s = \frac{\mathrm{J}_s}{\mathcal{W}} \sum_{n=1}^{N_r} \frac{1}{w_n}$$

2.1.3 Derivation of the BFIM for 2-D Maps

Consider a 2-D smoothed uniform map $f(\mathbf{p})$ with support $\mathcal{R} \subset \mathbb{R}^2$; with the assumptions (a) and (b) of [1, Sec II.B] and the notation introduced there, the pdf can be expressed as:

$$f(\mathbf{p}) = \frac{1}{\mathcal{A}} \sum_{i=1}^{N(y)} s\left(\frac{x - w_{m,i}(y)}{w_i(y)} \right) \cdot \sum_{j=1}^{N(x)} s\left(\frac{y - h_{m,j}(x)}{h_j(x)} \right)$$

where $s(\cdot)$ is a smoothing function as defined in Appedix 2.1.1. Also note that the regularity condition $\mathbb{E}_{\mathbf{p}} \left\{ \frac{\partial \ln f(\mathbf{p})}{\partial \mathbf{p}} \right\} = \mathbf{0}$ is easily verified thanks to the linear operators involved and $s(\cdot)$, which is assumed to respect that condition.

The first diagonal term of the BFIM associated to the prior knowledge 2 can be written, using the iterated expectation and focusing on the FI for the coordinate x, as:

$$[\mathbf{J}_{\mathbf{p}}]_{1,1} \triangleq \mathbb{E}_{\mathbf{p}} \left\{ \left(\frac{\partial \ln f(\mathbf{p})}{\partial x} \right)^2 \right\} = \mathbb{E}_y \left\{ \mathbb{E}_{x|y} \left\{ \left(\frac{\partial \ln f(\mathbf{p})}{\partial x} \right)^2 \right\} \right\} \tag{4}$$

If we now ignore the smoothing for the y coordinate, that is we make the approximation $f(\mathbf{p}) \approx \frac{1}{\mathcal{A}} \sum_{i=1}^{N(y)} s\left(\frac{x - w_{m,i}(y)}{w_i(y)} \right) = \frac{\mathcal{W}(y)}{\mathcal{A}} \frac{1}{\mathcal{W}(y)} \sum_{i=1}^{N(y)} s\left(\frac{x - w_{m,i}(y)}{w_i(y)} \right)$, where $\mathcal{W}(y) \triangleq \sum_{i=1}^{N(y)} w_i(y)$, we reduce the evaluation of the inner expectation to the evaluation of the FI of a 1-D map composed by $N_r = N(y)$ rectangles of widths $\{w_i(y)\}$ centered in the points $\{w_{m,i}(y)\}$. Thus, using the result obtained in Appendix 2.1.2, we have that:

$$\mathbb{E}_{x|y} \left\{ \left(\frac{\partial \ln f(\mathbf{p})}{\partial x} \right)^2 \right\} \approx \frac{\mathcal{W}(y)}{\mathcal{A}} \frac{\mathrm{J}_s}{\mathcal{W}(y)} \sum_{i=1}^{N(y)} \frac{1}{w_i(y)} \tag{5}$$

so that plugging Eq. (5) into Eq. (4) we obtain $[\mathbf{J_p}]_{1,1} \approx \frac{\mathrm{J}_s}{\mathcal{A}} \int_{\mathcal{Y}} \sum_{i=1}^{N(y)} \frac{1}{w_i(y)} dy$.

Symmetrically, ignoring the smoothing for the x coordinate, we obtain an approximated expression for the Bayesian Fisher Information (BFI) relative to the coordinate y; the two equations for x and y can be combined together as:

$$\operatorname{diag}\{\mathbf{J_p}\} = \frac{\mathrm{J}_s}{\mathcal{A}} \operatorname{diag}\left\{ \int_{\mathcal{Y}} \sum_{n=1}^{N(y)} \frac{dy}{w_n(y)}, \int \sum_{n=1}^{N(x)} \frac{dx}{h_n(x)} \right\} \tag{6}$$

Note however that the two approximations previously mentioned, considered together are exact only for rectangular maps. Also note that the cross-terms $[\mathbf{J_p}]_{2,1}$ and $[\mathbf{J_p}]_{1,2}$ of the BFIM are non-zero if and only if the parameters x and y are independent (like in a 2-D rectangle); in general, because of the smoothing the independence doesn't hold but is typically weak, so that a good approximation for the BFIM is [1, Eq. (4)]:

$$\mathbf{J_p} \approx \frac{\mathrm{J}_s}{\mathcal{A}} \operatorname{diag}\left\{ \int_{\mathcal{Y}} \sum_{i=1}^{N(y)} \frac{dy}{w_i(y)}, \int_{\mathcal{X}} \sum_{i=1}^{N(x)} \frac{dx}{h_i(x)} \right\}$$

For a discussion of this result please refer to [1].

References

[1] F. Montorsi, S. Mazuelas, and G. M. Vitetta, "On the Impact of A-Priori Information on Localization Accuracy," in *Positioning Navigation and Communication (WPNC), 2012 9th Workshop on*, 2012.

[2] H. L. V. Trees, *Detection, Estimation and Modulation Theory, Part III: Radar-Sonar Processing and Gaussian Signals in Noise.* John Wiley & Sons, Inc, 2001.

[3] S. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory.* Prentice Hall, 1993, vol. I.